

Now prepare a wave-packet; set

$$a_1^+ = \int d^3k F_1(\vec{k}) a^+(\vec{k}) \quad \text{with} \\ F_1(\vec{k}) = e^{-\frac{(\vec{k}-\vec{k}_1)^2}{4\sigma^2}}$$

Now let's prepare an initial state of two particles

$$|\psi\rangle = \lim_{t \rightarrow -\infty} a_1^+(t) a_2^+(t) |0\rangle \quad \text{with } \vec{k}_1 \neq \vec{k}_2$$

We want to calculate the amplitude for them to evolve into the final state

$$|\psi'\rangle = \lim_{t \rightarrow \infty} a_1^+(t) a_2^+(t) |0\rangle \quad \text{with } \vec{k}_1 \neq \vec{k}_2'$$

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we can normalize the wave-packets so that $\langle \tilde{c}_i | \tilde{c}_i \rangle = \langle \tilde{F} | \tilde{F} \rangle = 1$

To get a feel for what is happening, let's calculate the difference $a_i^+(\infty) - a_i^-(\infty)$. It will be illuminating to calculate this in a way following Srednicki. We'll need the result $a(\vec{k}) = i \int d^3x e^{ikx} \overleftrightarrow{\partial_0} \phi(x)$ with $\partial_0 = \frac{\partial}{\partial t}$

$$\overleftrightarrow{F} \partial_0 g = F \partial_0 g - (\partial_0 F) g$$

\Rightarrow simple to derive; do so

$$a_i^+(\infty) - a_i^-(\infty) = \int_{-\infty}^{\infty} dt \partial_0 a_i^+(t)$$

$$= -i \int d^3k F_i(\vec{k}) \underbrace{\int d^4x}_{\text{Sat}} \partial_0 [e^{-ikx} \overleftrightarrow{\partial_0} \phi(x)]$$

$$= i \int d^3k F_i(\vec{k}) \underbrace{e^{-ikx}}_{\int d^4x} \left\{ \partial_0^2 \phi + \cancel{\omega^2 \phi} \right\}$$

$$= i \int d^3k F_i(\vec{k}) \underbrace{e^{-ikx}}_{\int d^4x} \left\{ \partial_0^2 - \cancel{\nabla^2 + m^2} \right\} \phi$$

I.B.P on the spatial term

$$\int d^3x \left\{ \nabla^2 e^{ikx} \right\} \phi = - \int d^3x (\nabla e^{ikx}) \cdot \nabla \phi + \{ \phi \nabla e^{ikx} \}$$

where $|x| \rightarrow \infty$ on the surface

Tricky term is the surface term; let's see what happens

$$\Rightarrow \{ \phi(x) \nabla [\underbrace{\{ d^3 k f_1(\vec{k}) e^{-i\vec{k} \cdot \vec{x}} \}}_{\text{surface}}] \}$$

study this piece

$$\int d^3 k e^{-i(\vec{k}-\vec{k}_1)^2/4\sigma^2 - i\vec{k}_1 \cdot \vec{x}} = \int d^3 k e^{-i\vec{k}_1 \cdot \vec{x} - \vec{k}_1^2/4\sigma^2}$$

Shift $\vec{k} \rightarrow \vec{k} + \vec{k}_1 - 2i\sigma^2 \vec{x}$

$$= e^{-i\vec{k}_1 \cdot \vec{x} - \vec{k}_1^2/4\sigma^2} \int d^3 k e^{-\vec{k}^2/4\sigma^2}$$

On the surface, $|\vec{x}| \rightarrow \infty \Rightarrow e^{-\vec{\sigma}^2 \vec{x}^2} \rightarrow 0 \Rightarrow$ wave packet assures that the surface term vanishes

$$\begin{aligned} \Rightarrow a_1^+ (+\infty) - a_1^+ (-\infty) &= -i \int d^3 k f_1(\vec{k}) \left(\int d^4 x e^{-i\vec{k} \cdot \vec{x}} \{ \partial^2 + m^2 \} \right) \phi \\ &= -i \int d^3 k f_1(\vec{k}) \left(\int d^4 x e^{-i\vec{k} \cdot \vec{x}} \{ \partial^2 + m^2 \} \right) \phi(x) \end{aligned}$$

For the simple K.G. theory, $\{ \partial^2 + m^2 \} \phi(x) = 0$

\Rightarrow no change in operator, not surprising as $a^+(\vec{k})$ is t-independent. No interesting scattering \Rightarrow free

theory

But suppose we studied a more complicated theory, such as

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 + \frac{1}{6} g \phi^3$$

$$\Rightarrow \text{then } \{\partial^2 + m^2\} \phi = \frac{1}{2} g \phi^2, \text{ and } a_1^+(+\infty) - a_1^-(-\infty) \neq 0$$

\Rightarrow will have scattering in the interacting theory & this is what we'll be interested in

Go back to our scattering amplitude:

$$\langle F_{li} \rangle = \langle 0 | a_1'(+\infty) a_2'(+\infty) a_1^-(-\infty) a_2^-(-\infty) | 0 \rangle$$

Note that the later time $(+\infty)$ is to the left, while the earlier time $(-\infty)$ is to the right. Introduce the time-ordering symbol T that performs this operation

$$\langle F_{li} \rangle = \langle 0 | T \{ a_1'(+\infty) a_2'(+\infty) a_1^-(-\infty) a_2^-(-\infty) \} | 0 \rangle$$

Let's see what this does. Replace

$$a_2^-(-\infty) = a_2^+(+\infty) + i \int d^3 k f_i(\vec{k}) \int d^4 x e^{ikx} \{\partial^2 + m^2\} \phi(x)$$

The T symbol, in the first term with $a_2^-(+\infty)$, does the following

$$\langle 0 | T \{ a_1'(+\infty) a_2'(+\infty) a_1^-(-\infty) a_2^-(+\infty) \} | 0 \rangle$$

$$\Rightarrow \langle 0 | T \{ a_1'(+\infty) a_2'(+\infty) a_2^+(+\infty) a_1^-(+\infty) \} | 0 \rangle$$

Keep track of all terms, systematically replacing

$$a_{1,2}^+(-\omega) = a_{1,2}^+(+\omega) + i \int d^3k F_1(\vec{k}) \left(d^4x e^{-ikx} \{ \partial_x^2 + m^2 \} \phi(x) \right)$$

$$a_{1,2}^+(+\omega) = a_{1,2}(-\omega) + i \int d^3k F_1(\vec{k}) \left(d^4x e^{+ikx} \{ \partial_x^2 + m^2 \} \phi(x) \right)$$

The wave-packets can be removed at this point

$$F_1(\vec{k}) = \delta^{(3)}(\vec{k} - \vec{k}_1) \Rightarrow \int d^3k F_1(\vec{k}) = 1$$

Five types of terms:
(a) all 4 a 's remain;
(b) 3 a 's remain
(c) 2 a 's remain; all 1 a remains
(d) none remain

$$(a) \langle 0 | T \{ a_1'(-\omega) a_2'(-\omega) a_1^+(+\omega) a_2^+(+\omega) \} | 0 \rangle$$

$$\Rightarrow \langle 0 | T \{ a_1^+(+\omega) a_2^+(+\omega) a_1'(-\omega) a_2'(-\omega) \} | 0 \rangle = 0$$

since $a(\vec{k})|0\rangle = 0$

(b) Example of such a term is

$$+ i \int d^4x_1 e^{-ik_1 x_1} (\partial_x^2 + m^2) \langle 0 | T \{ a_1'(-\omega) a_2'(-\omega) \phi(x_1) a_2^+(+\omega) \} | 0 \rangle$$

kills vacuum, $= 0$

Others clearly vanish also

(c) Example of such a term is

$$+ i \Gamma^2 \int d^4x_1 d^4x_2 e^{-ik_1 x_1 - ik_2 x_2} \langle 0 | T \{ a_1'(-\omega) \phi(x_2) \phi(x_1) a_2^+(+\omega) \} | 0 \rangle$$

$(\partial_x^2 + m^2)(\partial_x'^2 + m^2) = 0$

It should be clear that the only surviving term is with all $a_i a_i^*$ removed

$$\langle \text{CF} | i \rangle = (t_i)^4 \left(d^4 x_1 d^4 x_2 d^4 x_1' d^4 x_2' (\partial_1^2 + m^2) (\partial_2^2 + m^2) \right. \\ \left. (\partial_1'^2 + m^2) (\partial_2'^2 + m^2) \langle 0 | T\{\phi(x_1)\phi(x_2) \right. \\ \left. - ik_1 x_1 - ik_2 x_2 + ik_1' x_1' + ik_2' x_2' \phi(x_1')\phi(x_2') \} | 0 \rangle \right)$$

LSZ reduction formula

It is clear that this can be generalized to SREDNICKI. Although the time-ordering symbol appears artificial, we'll see that it arises quite naturally when we consider path integrals.

I have to be careful about a few issues in our derivation. We used a free field theory expression for $a_i(t)$; what happens in an interacting theory? Does $a_i(t)$ still produce a properly normalized, single-particle state? Let's check.

First recall the time evolution of a quantum operator

$$\phi(\vec{x}, t) = e^{iHt} \phi(\vec{x}, 0) e^{-iHt}$$

It will come as no surprise that the relativistic generalization of this is $\phi(x) = e^{i\vec{P} \cdot \vec{x}} \phi(0) e^{-i\vec{P} \cdot \vec{x}}$

Study the overlap of $a_i^*(t) | 0 \rangle$ with $| 07, 1\rangle$, multi-particle states

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$$\begin{aligned}
 @) \langle 0 | a_i^+(t) | 0 \rangle &= -i \int d^3 k F_i(\vec{k}) \left(\int d^3 x e^{-ik \cdot x} \overset{\leftrightarrow}{\partial}_0 \right) \underbrace{\langle 0 | \phi(x) | 0 \rangle}_{\langle 0 | \phi(0) | 0 \rangle} \\
 &= \int d^3 k F_i(\vec{k}) \left(\int d^3 x k^0 e^{-ik \cdot x} \langle 0 | \phi(0) | 0 \rangle \right) \quad \text{since } e^{i k^0 x} | 0 \rangle = | 0 \rangle \\
 &= (2\pi)^3 \left(\int d^3 k F_i(\vec{k}) k^0 \delta^{(3)}(\vec{k}) \right) e^{-imt} \langle 0 | \phi(0) | 0 \rangle \\
 &= (2\pi)^3 m e^{-imt} F_i(0) \langle 0 | \phi(0) | 0 \rangle
 \end{aligned}$$

For this to vanish (no overlap of 0+1-particle states),
 $\langle 0 | \phi(0) | 0 \rangle = 0 \Rightarrow$ if not, need to shift the
Field $\phi(x) \rightarrow \phi(x) + \checkmark$ to make true (i.e., for the Higgs)

(b) Now a one particle state

$$\langle p | a_i^+(t) | 0 \rangle = -i \int d^3 k F_i(\vec{k}) \left(\int d^3 x e^{-ik \cdot x} \overset{\leftrightarrow}{\partial}_0 \right) \langle p | \phi(x) | 0 \rangle$$

$$\text{Set } \langle p | \phi(x) | 0 \rangle = e^{ip \cdot x} \langle p | \phi(0) | 0 \rangle$$

$$= -i \int d^3 k F_i(\vec{k}) \left(\int d^3 x \left\{ \overbrace{e^{-ik \cdot x} \overset{\leftrightarrow}{\partial}_0}^{i(p^0 + k^0)} e^{ip \cdot x} \right\} \langle p | \phi(0) | 0 \rangle \right)$$

$$= i(p^0 + k^0) \int d^3 k F_i(\vec{k}) e^{i(p-k) \cdot x} \langle p | \phi(0) | 0 \rangle$$

x integral gives $(2\pi)^3 \delta^{(3)}(\vec{p} - \vec{k})$, sets $p^0 = k^0 = \omega$

$$= 2\omega (2\pi)^3 F_i(\vec{p}) \langle p | \phi(0) | 0 \rangle$$

For $F_i(\vec{p}) = \delta^{(3)}(\vec{p} - \vec{h}_i)$, we get a plane wave; need $\langle p | \phi(0) | 0 \rangle = 1$ for proper normalization. We'll later need wave-function renormalization to impose this

(c) Let's look at the overlap with multi-particle states.

Form one via $|Y\rangle = \sum_n \left\{ d^3 p \psi_n(\vec{p}) |p, n\rangle \right\}$

\uparrow symbolic; sums over momenta between particles, etc.

$p = \text{total momentum of all particles}$

$$\begin{aligned} \langle 4 | a_i^\dagger(t) | 0 \rangle &= -i \left(d^3 k F_i(\vec{k}) \right) \left(d^3 x \sum_n \left\{ d^3 p \psi_n^*(\vec{p}) \right. \right. \\ &\quad \left. \left. \left\{ e^{-ik \cdot x} \overleftrightarrow{\partial} \underbrace{\langle p, n | \phi(x) | 0 \rangle}_{e^{ip \cdot x}} \right\} \right. \right. \\ &\quad \left. \left. \left. \langle p, n | \phi(0) | 0 \rangle \right\} \right) \\ &= \left(d^3 k F_i(\vec{k}) \right) \left(d^3 x \sum_n \left\{ d^3 p (p^0 + k^0) e^{i(p^0 - k^0)t} e^{i(\vec{p} - \vec{k}) \cdot \vec{x}} \right. \right. \\ &\quad \left. \left. \left. \langle p, n | \phi(0) | 0 \rangle \right\} \right) \\ &= (2\pi)^3 \int d^3 p F_i(\vec{p}) \sum_n (p^0 + k^0) e^{i(p^0 - k^0)t} \langle p, n | \phi(0) | 0 \rangle \\ &\quad \text{with } k^0 = \sqrt{\vec{p}^2 + m^2} \end{aligned}$$

We need to know a little bit about multi-particle states, in particular their energy

Consider a two-particle state. First, $p^u = p_1^u + p_2^u$. Look at it in the frame where $\vec{p} = \vec{p}_1 + \vec{p}_2 = 0 \Rightarrow$ they're moving in a back-to-back direction.

$$p^2 = (p_1^0 + p_2^0)^2 \Rightarrow \text{without any bound-state effects, } p_1^0 \geq m, p_2^0 \geq m$$

$$\Rightarrow p^2 \geq (2m)^2$$

$$\text{Set } p^2 \equiv M^2$$

In the frame we care about, $p^0 = \sqrt{\vec{p}^2 + M^2}$ and $p^0 \geq k^0$
since $M > m$

We care about $\langle \psi | a_i^{+}(\pm\omega) | 0 \rangle$. As $t \rightarrow \pm\infty$, $e^{i(p^0 - k^0)t}$ oscillates very quickly \Rightarrow integral over \vec{p} vanishes. No overlap with multi-particle states. $a_i^{+}(\pm\omega)$ creates the appropriate 1-particle state as long as

$$\langle 0 | \phi(0) | 0 \rangle = 0$$

$$\langle p | \phi(0) | 0 \rangle = 1$$

Path integrals in QM

Computation of the time-ordered products that appear in LSZ is most naturally done using path integrals. Let's review them in Q.M. Let's work with the Hamiltonian $H = \frac{p^2}{2m} + V(Q)$

Time evolution of a Schrödinger picture state:

$$i \frac{d}{dt} |\Psi(t)\rangle = H|\Psi(t)\rangle \Rightarrow i \frac{d}{dt} \langle q | \Psi(t) \rangle = \langle q | H |\Psi(t)\rangle$$

Now switch to Heisenberg picture, where the position operator $Q \rightarrow Q(t) \rightarrow$ its eigenstates $|q\rangle$ become time-dependent, $|q\rangle$ time-independent

$$i \frac{d}{dt} \langle q, t | \Psi \rangle = \langle q, t | H | \Psi \rangle \Rightarrow i \frac{d}{dt} \langle q, t | = \langle q, t | H$$

$$\text{so that } \langle q, t | = \langle q, 0 | e^{-iHt} = \langle q | e^{-iHt}$$

The amplitude to go from $|q', t'\rangle$ to $|q'', t''\rangle$ is then

$$\langle q'', t'' | q', t' \rangle = \langle q'' | e^{-iH(t'' - t')} \underbrace{| q' \rangle}_{T}$$

The completeness relation for position eigenstates is

$$\int dq |q\rangle \langle q| = 1$$

Divide the evolution from $t' \rightarrow t''$ into segments of time $\Delta t = \frac{T}{N+1}$ by inserting N complete sets of $|q\rangle$

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$$\langle q'' | e^{-iHt} | q' \rangle = \int_{j=1}^N d\alpha_j \langle q'' | e^{-iH\delta t} | q_N X_{q_N} | e^{-iH\delta t} | q_m \rangle \dots \langle q_i | e^{-iH\delta t} | q' \rangle$$

Split up the P, Q dependent in the Hamiltonian:

$$e^{-i\delta t} \left\{ \frac{P^2}{2m} + V(Q) \right\} = e^{-i\delta t \frac{P^2}{2m}} e^{-i\delta t V(Q)} e^{\underbrace{-\frac{1}{2}(-i\delta t)^2 \left[\frac{P^2}{2m}, V(Q) \right]}_{=1+O(\delta t)^2}}$$

Look at one of the intermediate amplitudes

$$\langle q_i | e^{-iH\delta t} | q_j \rangle = \langle q_i | e^{-i\delta t \frac{P^2}{2m}} e^{-i\delta t V(Q)} | q_j \rangle \quad \text{with } i=j+1$$

$$\begin{aligned} &= \int dp_j \langle q_i | e^{-i\delta t \frac{P^2}{2m}} | p_j X_{p_j} | e^{-i\delta t V(Q)} | q_j \rangle \\ &= \int dp_j e^{-i\delta t \frac{p_i^2}{2m}} e^{-i\delta t V(q_j)} \langle q_i | p_j X_{p_j} | q_j \rangle \end{aligned}$$

$$\text{Recall that } \langle q_i | p_i \rangle = \frac{1}{\sqrt{2\pi}} e^{ip_i q_i}$$

$$\text{(Derivation: } \langle \hat{p} | p \rangle = \langle p | p \rangle, \text{ while } \langle q | \hat{p} | p \rangle = \frac{\hbar}{i} \frac{\partial}{\partial p} \langle q | p \rangle \text{)}$$

$$\Rightarrow \langle q | p \rangle = N e^{ipq} \rightarrow \text{set } N \text{ via } = p \langle q | p \rangle$$

$$\left\{ \int dp \langle q_i | p X_{p_j} | q_j \rangle = \delta(q_i, q_j) \right\}$$

$$\rightarrow \int \frac{dp_j}{2\pi} e^{-i\delta t H(p_j, q_j)} e^{ip_j(q_i - q_j)} \uparrow_{q_j+1}$$

Combine all these

$$\langle q'' | e^{-iHt} | q' \rangle = \left(\prod_{k=1}^N dq_k \right) \left(\prod_{j=0}^N \frac{dp_j}{2\pi} \right) e^{i \sum_{j=0}^N \{ p_j(q_{j+1}-q_j) - \delta t H(p_j, q_j) \}}$$

with $q_0 = q'$
 $q_{N+1} = q''$

$$\text{Set } \delta t \rightarrow 0; \quad p_j \rightarrow p(t) \\ q_j \rightarrow q(t)$$

$$q_{j+1} - q_j \rightarrow \delta t \dot{q}(t)$$

$$\sum_{j=0}^N \delta t \rightarrow \int_{t'}^{t''} dt$$

$$\langle q'', t'' | q', t' \rangle = \left(Dq \right) \left(Dp \right) e^{i \int_{t'}^{t''} dt \{ p(t) \dot{q}(t) - H(p(t), q(t)) \}}$$

Defined as $\delta t \rightarrow 0$ limit of above; symbolic
Functional integration

We can evaluate this for the Hamiltonian we're considering.
In the discrete case, all the p_j integrals are of the form

$$\int dp_j e^{i \delta t F(p_j)}, \quad F(p_j) \text{ is quadratic in } p_j$$

\Rightarrow Define p' such that $F'(p') = 0 \Rightarrow$ expand then

$$F(p_j) = F(p') + \frac{1}{2} (p_j - p')^2 F''(p') \quad (\text{exact!})$$

$$\Rightarrow e^{i \delta t F(p')} \int_{-\infty}^{\infty} dp_j e^{-\frac{i}{2} \delta t (p_j)^2 F''(p')} \xrightarrow{\int \frac{d\pi}{\sqrt{i \delta t F''(p')}}} e^{i \delta t F(p')}$$

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The equation to determine \dot{q}' is

solve for \dot{p}' in
terms of \dot{q}_i, \ddot{q}_i

$$\frac{dF}{dp_j} \Big|_{p_j = p'} = 0 \Rightarrow \frac{\dot{q}_j + \ddot{q}_j}{\delta t} - \frac{\partial H}{\partial p_j} = 0 \Rightarrow \dot{q}'_j = \frac{\partial H(p', q'_j)}{\partial p_j}$$

\Rightarrow Legendre transform from $p' \rightarrow \dot{q}' \Rightarrow \frac{p' \dot{q}' - H}{m} = L$

The $F''(p')$ factor is just a constant, $\frac{1}{m}$

$$\Rightarrow \langle q'' | e^{-iHT} | q' \rangle = \left(\frac{1}{2\pi} \sum_{k=1}^N \frac{1}{2\pi} \sqrt{\frac{2\pi m}{\delta t}} \right) e^{-i \sum_{j=0}^N L(q_j, \dot{q}_j)}$$

$\frac{1}{2\pi} \sqrt{\frac{2\pi m}{\delta t}}$ Define as $\int Dq$

$$= \int Dq e^{i \int_{t'}^{t''} dt L[\dot{q}(t), q(t)]} = \int Dq e^{iS}$$

This is the Feynman path integral. In classical mechanics, only get contributions with $\delta S = 0$. In quantum, all paths contribute.

An interesting thing happens when we start inserting operators inside the transition amplitude.

$\langle q'', t'' | Q(t_1) | q', t' \rangle \Rightarrow$ go to Schrödinger picture

$$\begin{aligned} &= \langle q'' | e^{-iHt''} e^{iHt_1} Q e^{-iHt_1} e^{iHt'} | q' \rangle \\ &= \left\langle \int d\alpha_i q'' | e^{-iH(t''-t_1)} Q | \alpha_i \right\rangle \underbrace{\chi_{\alpha_i}(t_1) e^{-iH(t_1-t')} | q' \rangle} \\ &= \left\langle \int d\alpha_i \alpha_i \langle q'' | e^{-iH(t''-t_1)} | \alpha_i \right\rangle \chi_{\alpha_i}(t_1) e^{-iH(t_1-t')} | q' \rangle \end{aligned}$$

Now go through same derivation as before, only difference is $\alpha_i \rightarrow \alpha(t_1)$ in the front

$$\Rightarrow \langle q'', t'' | Q(t_1) | q', t' \rangle = \left\langle \int Dq q(t_1) e^{i \int_{t_1}^{t'} dt L [q(t), q(H)]} \right\rangle$$

Suppose we have $\langle q'', t'' | Q(t_2) Q(t_1) | q', t' \rangle$

IF $t_2 > t_1$, we can write this as

$$\langle q'' | e^{-iH(t''-t_2)} Q e^{-iH(t_2-t_1)} Q e^{-iH(t_1-t')} | q' \rangle$$

and Find $\left\langle \int Dq q(t_2) q(t_1) e^{i \int_{t_1}^{t_2} dt L [q(t), q(H)]} \right\rangle$

This doesn't work if $t_2 < t_1$; we get

$$\langle q'' | e^{-iH(t''-t_2)} Q e^{+iH|t_2-t_1|} Q e^{-iH(t_1-t')} | q' \rangle$$

→ doesn't have the same sign for δt in the middle,
doesn't turn into $\int dt$

→ Naturally gives the time ordering symbol

$$\langle q'', t'' | \{Q(t_2) Q(t_1)\} | q', t' \rangle = \int Dq q(t_1) q(t_2) e^{iS}$$

This is exactly what we need to compute particle scattering according to LSZ. We just need to clean up a few more things to make this directly useable.

First, for the discrete q_i , it is clear that

$$\frac{\partial q_i}{\partial \alpha_j} = \delta_{ij} \quad \text{Introduce a functional derivative that is the continuum analog of this}$$

$$\frac{\partial}{\partial q(t_1)} q(t_2) = \delta(t_2 - t_1)$$

Modify the Lagrangian to $L[q^i(t), \dot{q}(t)] \rightarrow L[\bar{q}(t), q(t)] + J(t)q(t)$

with J some currently arbitrary function

$$\langle q'', t'' | q', t' \rangle_J = \int Dq e^{i \int_{t'}^{t''} dt [L(q, \dot{q}) + Jq]}$$

$$\text{Take } \left[\frac{1}{i} \frac{\partial}{\partial J(t_1)} \langle q'', t'' | q', t' \rangle_J \right]_{J=0}$$

$$= \int Dq \frac{1}{i} \left\{ i \int_{t'}^{t''} dt \dot{q}(t) \delta(t - t_1) \right\} e^{iS}$$

$$= \int Dq(t) e^{iS} = \langle q'', t'' | Q(t_i) | q', t' \rangle$$

→ we can generate time-ordered products with functional derivatives

$$\langle q'', t'' | T\{Q(t_1) \dots Q(t_n)\} | q', t' \rangle$$

$$= \left[\frac{\partial}{\partial J(t_1)} \dots \frac{1}{i} \frac{\partial}{\partial J(t_n)} \langle q'', t'' | q', t' \rangle \right]_{J=0}$$

We'll also be interested in so-called vacuum-to-vacuum transitions in the presence of a source (recall our LSZ reduction formula had $\langle 0 | T\{Q(x_1) \dots Q(x_n)\} | 0 \rangle$). We'll use a complex-plane trick to arrange this:

$$\langle q'', t'' | = \langle q'' | e^{-iHt''} \Rightarrow$$

$$\Rightarrow \langle q'' | e^{-i(H - i\epsilon)t''}$$

Insert a complete set of energy eigenstates

$$= \sum_n \langle q'' | e^{-i(H - i\epsilon)t''} | n \rangle_n = \sum_n \langle q'' | n \rangle_n e^{-i(E_n - i\epsilon)t''}$$

Now take $t'' \rightarrow +\infty$, $\epsilon \ll 1$. If the theory has a good ground state, then E_0 is bounded below \Rightarrow shift so that $E_0 = 0$. Then, this limit picks out the $|0\rangle$ state; all others have $e^{-\epsilon E_n t''} \rightarrow 0$

$$\lim_{t'' \rightarrow +\infty} \langle q'', t'' | = \langle q'' | 0 \rangle \langle 0 |$$

Similarly, For $\langle a' | t' \rangle$, taking $t' \rightarrow -\infty + i\epsilon$ gives

$$\langle a' | q' \rangle \propto \langle a' | 0 \rangle$$

This is equivalent to taking $H \rightarrow (t\epsilon)H$ (check this!)

$$\begin{aligned} \Rightarrow \langle 0 | 0 \rangle_J &= \lim_{\substack{t'' \rightarrow \infty \\ t' \rightarrow -\infty}} \frac{\langle a'', t'' | a' | t' \rangle}{\langle a'' | 0 \rangle \langle 0 | a' \rangle} \quad \text{just some constant,} \\ &\qquad \qquad \qquad \text{absorb into definition of } D \\ &= \int Dp Dq e^{i \int_0^\infty dt [p\dot{q} - (1-i\epsilon)H + J_q]} \end{aligned}$$

We can again go the the Lagrangian here, being sure to keep the $i\epsilon$ - $i \int_0^\infty dt \{ L[q, \dot{q}; \epsilon] + \bar{J}_q \}$ keep track of

$$\langle 0 | 0 \rangle_J = \int Dq e^{-i \int_0^\infty dt \{ L[q, \dot{q}; \epsilon] + \bar{J}_q \}}$$

One more vitally important thing. Suppose

$$L[q, \dot{q}; \epsilon] = L_0[q, \dot{q}] + L_1[q] \quad (\text{drop } \epsilon \text{ for right now})$$

where L_1 is small somehow (has a coupling g^2)
 L_1 is also hard to deal with (nonlinear, for example,

We can imagine expanding in L_1

$$e^{-i \int_0^\infty dt \{ L_0 + \bar{J}_q + L_1 \}} = \left\{ 1 + i \int_0^\infty dt L_1(q) \dots \right\} e^{-i \int_0^\infty dt [L_0 + \bar{J}_q]}$$

↑
will be something like $g^{q^3(t)}$, i-

\Rightarrow We have things like $\int Dq \left(\int_{-\infty}^{\infty} dt' q_1^3(t') e^{i \int_{-\infty}^t dt' (L_0 + J_0)} \right)$

replace by $\left(\frac{1}{i} \frac{\partial}{\partial J(t)} \right)^3$

$\Rightarrow \langle 0|0|0|J \rangle = \int Dq e^{i \int_{-\infty}^{\infty} dt L_i \left[\frac{1}{i} \frac{\partial}{\partial J(t)} \right]} e^{i \int_{-\infty}^{\infty} dt \{ L_0(q_i) + J q_i \}}$

where we expand this as high as possible